On Artinian Rings with Restricted Class of Injectivity Domains

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Abstract

In a recent paper of Alahmadi, Alkan and López–Permouth, a ring R is defined to have no (simple) middle class if the injectivity domain of any (simple) R–module is the smallest or largest possible. Er, López–Permouth and Sökmez use this idea of restricting the class of injectivity domains to classify rings, and give a partial characterization of rings with no middle class. In this work, we continue the study of the property of having no (simple) middle class. We give a structural description of right Artinian right nonsingular rings with no right middle class. We also give a characterization of right Artinian rings that are not SI to have no middle class, which gives rise to a full characterization of rings with no middle class. Furthermore, we show that commutative rings with no middle class are those Artinian rings which decompose into a sum of a semisimple ring and a ring of composition length two. Also, Artinian rings with no simple middle class are characterized. We demonstrate our results with several examples.

1 Introduction

Throughout this paper, our rings will be associative rings with identity, and modules will be unital right modules, unless otherwise stated. For any ring R, Mod–R will denote the category of all right R–modules.

Let R be a ring. Recall that an R–module M is *injective relative to an R–module N* (or, *M is N–injective*) if, for any submodule K of N, any R –homomorphism $f : K \to M$ extends to some member of $Hom_R(N, M)$. It is evident that every module is injective relative to semisimple modules. Thus, for any R–module M, *the injectivity domain* $\mathfrak{In}^{-1}(M) = \{N \in \text{Mod–}R : M \text{ is } N\text{–injective}\}$ of M contains all semisimple right R–modules. In [1], Alahmadi, Alkan and López–Permouth initiated the study of *poor modules,* namely modules whose injectivity domains consist only of semisimple modules in Mod–R. They consider rings over which every right module is either injective or poor, and refer such rings as *having no right middle class.* The study of rings with no middle class has a growing interest in recent years (see [1], [3], [6], and [9]).

In [3], Er, López–Permouth and Sökmez studied the rings with no right middle class and gave a partial characterization of such rings. The following two theorems summarize the results on rings with no right middle class obtained in [3]. To simplify the statements, we assume that the ring R is not semisimple Artinian. All statements can be made to fit that possibility by setting $T = 0$.

Theorem 1 Let R be a right SI–ring. Then R has no right middle class if and only if $R \cong S \oplus T$, where S is semisimple Artinian and T is either

    (i) Morita equivalent to a right PCI–domain or

    (ii) an indecomposable ring with homogenenous essential right socle satifying one of the following equivalent conditions (where Q is the maximal right quotient ring of T):

        (a) Non-semisimple quasi–injective right T–modules are injective.

        (b) Proper essential submodules of Q_T are poor.

        (c) For any submodule A of Q_T containing $Soc(T_T)$ properly, $QA = Q$.

Those rings of type (ii) are either right Artinian or right V–rings and have a unique simple singular right T–module up to isomorphism.

Theorem 2 Let R be a ring with no right middle class which is not right SI. Then $R \cong S \oplus T$, where S is semisimple Artinian and T is an indecomposable right Artinian ring satisfying the following conditions:

(i) $soc(T_T) = Z(T_T) = J(T)$,

(ii) T has homogeneous right socle, and

(iii) there is a unique non–injective simple right T–module up to isomorphism.

In this case T is either a QF–ring with $J(T)^2 = 0$, or poor as a right module. Conversely, if T is a QF–ring with homogeneous right socle and $J(T)^2 = 0$, then T has no right middle class.

Note that the authors of [3] could not reverse this implication to show that the conditions (i)–(iii) in Theorem 2 above are sufficient as well as necessary. As a matter of fact, we show in our work that there exist rings satisfying conditions (i)–(iii) in Theorem 2 above which are poor as a right module over itself and do have right middle class (see Examples 2.22 and 2.23). We also give a complete characterization of non–SI–rings with no right middle class (see Theorem 2.16).

A characterization of right Artinian rings with no right middle class was given in Corollary 3.2 of [9]. Using that result in conjunction with those of [3], we have the following two complete characterizations:

Theorem 3 Let R be any ring. Then R has no right middle class if and only if $R \cong S \oplus T$, where S is semisimple Artinian and T satisfies one of the following conditions:

(i) T is Morita equivalent to a right PCI–domain, or

(ii) T is a right SI right V–ring with the following properties:

(a) T has essential homogeneous right socle and

(b) for any submodule A of Q_T which does not contain the right socle of T properly, $QA = Q$, where Q is the maximal right quotient ring of T, or

(iii) T is a right Artinian ring whose Jacobson radical properly contains no nonzero ideals.

Theorem 4 Let R be any ring. Then R has no right middle class if and only if $R \cong S \oplus T$, where S is semisimple Artinian and T satisfies one of the following conditions:

(i) T is Morita equivalent to a right PCI–domain, or

(ii) (a) T is a right Artinian ring or a right SI right V–ring with homogeneous essential right socle, and

(b) every nonsemisimple quasi–injective right T–module is injective.

In the process of studying these rings, various necessary or sufficient conditions are presented in [3]. For instance,

(P1) R has homogeneous right socle, and

(P2) there is a unique simple singular right R–module up to isomorphism

are necessary conditions for a nonsemisimple indecomposable right SI–ring R to have no right middle class (see [3, Theorem 2]). Likewise, in [3, Proposition 6], it is shown that right Artinian right SI–rings with homogeneous right socle anda unique local module of length two up tp isomorphism must have no right middle class. We show here that **(P1)** and **(P2)** are not sufficient while the condition that the ring has a unique local module of length two up to isomorphism is not necessary (see Examples 2.9(i) and 2.13).

It is shown, in [3, Corollary 5], that if R is an indecomposable right nonsingular right Artinian ring with no right middle class, then R is isomorphic to a formal triangular matrix ring of the form $\begin{pmatrix} S & 0 \\ A & S' \end{pmatrix}$, where S and S' are simple Artinian rings and A is an S'–S–bimodule. Using the theory of Morita equivalences, we see that such rings simplifies to formal triangular matrix rings of the form $\begin{pmatrix} D & 0 \\ \mathbb{M}_{n \times 1}(D) & D' \end{pmatrix}$, where D is a division ring and D' is a division subring of $\mathbb{M}_n(D)$ for some positive integer n (Theorem 2.1). We also prove that certain conditions on D' characterizes these triangular rings to have no right middle class which yields a general characterization for right nonsingular right Artinian rings to have no right middle class (Theorem 2.5). This result also enables us to produce many interesting examples of right nonsingular right Artinian rings with no right middle class.

It is also known from [3, Corollary 6] that if R is an indecomposable right Artinian ring with no right middle class which is not right SI, then R is isomorphic to a formal triangular matrix ring of the form $\begin{pmatrix} \mathbb{M}_n(A) & 0 \\ X & B \end{pmatrix}$, where A is a (nonsemisimple) local right Artinian ring, B is a semisimple Artinian ring, and X is a B–$\mathbb{M}_n(A)$–bimodule. As a matter of fact, a right Artinian ring which is not right SI has no right middle class if and only if $R \cong S \oplus \mathbb{M}_n(A)$ where S is semisimple Artinian and A is a local right Artinian ring whose Jacobson radical properly contains no nonzero ideals (Theorem 2.16).

In Section 3, we restrict our attention to only simple modules, and consider rings whose simple right modules are either injective or poor. Such rings are said to have no simple middle class (see [1]). We give necessary and sufficient conditions for a right Artinian ring to have no simple middle class.

The last section of our paper is concerned with the property of having no (simple) middle class in the commutative setting. We give a complete description of commutative rings with no middle class. In particular, we see that a commutative ring with no middle class is Artinian. We conclude our work with a characterization of commutative Noetherian rings to have no simple middle class.

Recall that a ring is said to be a right V–ring if every simple right module is injective. As a generalization of right V–rings, right GV–rings were introduced by Ramamurthi and Rangaswamy in [10]. A ring is called right GV if every simple singular module is injective, or equivalently, every simple module is either injective or projective. We call a ring right SI if every singular right module is injective (see[2]). Note that semilocal right GV–rings are right SI.

If M is an R–module, then $E(M)$, $J(M)$, $Z(M)$ and $Soc(M)$ will respectively denote the injective hull, Jacobson radical, the singular submodule and the socle of M. We will use the notations $\leq$ and $\leq_e$ in order to indicate submodules and essential submodules, respectively. For a module with a composition series, $cl(M)$ stands for the composition length of M. The ring of $n \times n$ matrices over a ring R will be denoted by $\mathbb{M}_n(R)$. The notation $A[i,j]$ will be used to indicate the (i,j)–th entry of a matrix A. We use e_{ij} to designate the standard matrix unit of $\mathbb{M}_n(R)$ with 1 in the (i,j)–th entry and zeros elsewhere. For any unexplained terminology, we refer the reader to [4] and [8].

2   Artinian Rings with No Middle Class

In [3, Corollary 5], Er, López–Permouth, and Sökmez proved that if R is an indecomposable right nonsingular right Artinian ring with no right middle class, then R is isomorphic to a formal triangular matrix ring of the form $\begin{pmatrix} S & 0 \\ A & S' \end{pmatrix}$, where S and S' are simple Artinian rings and A is an S'–S–bimodule. With the following theorem, we see that to determine when such rings have no right middle class, it is enough to consider formal triangular matrix rings of the much simpler form $\begin{pmatrix} D & 0 \\ \mathbb{M}_{n \times 1}(D) & D' \end{pmatrix}$, where D is a division ring and D' is a division subring of $\mathbb{M}_n(D)$ for some positive integer n

Theorem 2.1 *If R is a right Artinian right SI ring satisfying the properties $(P1)$ and $(P2)$, then it is Morita equivalent to a formal triangular matrix ring of the form*

$$\begin{pmatrix} D & 0 \\ \mathbb{M}_{n \times 1}(D) & D' \end{pmatrix},$$

where D is a division ring and D' is a division subring of $\mathbb{M}_n(D)$.

Proof. Since R is a right Artinian ring, there exists a complete set $\{e_1, \ldots, e_l, f_1, \ldots, f_m\}$ of local orthogonal idempotents such that the $e_i R$'s are simple and $f_i R$'s are nonsemisimple local. Then $R = e_1 R \oplus \cdots \oplus e_l R \oplus f_1 R \oplus \cdots \oplus f_m R$. By $(P2)$, $f_i R \cong f_j R$ for all i, j. Moreover, since $e_i R$ is nonsingular, there is no nonzero R–homomorphism from $f_j R$ into $e_i R$, and so $e_i R f_j = 0$ for all i, j. As R is right SI, $R/Soc(R_R)$ is semisimple Thus, $J(R) \leq Soc(R_R)$. This gives that $J(f_i R) = Soc(f_i R)$. We set $e = e_1 + f_1$. Then $eR = e_1 R \oplus f_1 R$. We shall prove that e is a full idempotent of R, i.e., $ReR = R$. It is clear that $e_1, f_1 \in ReR$. Since R has homogeneous right socle, $e_i Re_1 R \neq 0$ for all $i = 1, \ldots, l$. Then $e_i \in e_i R = e_i Re_1 R \leq ReR$ for all $i = 1, \ldots, l$. Now, assume that $f_k \notin f_k R f_1 R$. Then $f_k R f_1 R \leq Soc(f_k R)$. Since R has homogeneous right socle and $e_i R f_j = 0$ for all i, j, $f_k R f_1 R f_1 = 0$. Let $u : f_1 R \to f_k R$ be an isomorphism of right R–modules. Then $u(f_1 R f_1 R f_1) = u(f_1) R f_1 R f_1 \leq f_k R f_1 R f_1 = 0$, and so $f_1 \in f_1 R f_1 R f_1 = 0$, a contradiction. It follows that $f_k \in f_k R f_1 R \leq ReR$ for all $k = 1, \ldots, m$. Therefore, $ReR = R$.

Now let α be an endomorphism on $E(f_1 R)$, and let α' be the restriction of α to $Soc(f_1 R)$. Suppose $Soc(f_1 R) = B_1 \oplus \cdots \oplus B_n$, where $B_1, \ldots, B_n$ are simple right R–modules isomorphic to $e_1 R$, and let $v_k : B_k \to e_1 R$ be an isomorphism. Set $\alpha_k = i_k v_k^{-1}$ and $\beta_k = v_k \pi_k$, where i_k is the natural embedding of B_k into $Soc(f_1 R)$, and π_k the natural projection of $Soc(f_1 R)$ onto B_k. Then the correspondence

$$\alpha \xleftrightarrow{\varphi} (\beta_i \alpha' \alpha_j)_{n \times n}$$

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between $End(E(f_1R))$ and $\mathbb{M}_n(End(e_1R))$ gives a ring isomorphism. Moreover, since $E(f_1R)$ is nonsingular and $E(f_1R)/Soc(f_1R)$ is singular, this correspondence yields an embedding of $End(f_1R)$ into $\mathbb{M}_n(End(e_1R))$. We denote $\varphi(End(f_1R))$ by D'. Note that D' is a division ring in view of the proof of [3, Corollary 5]. Now it is routine to check that the mapping

$$eRe \cong \begin{pmatrix} End(e_1R) & 0 \\ Hom(e_1R, f_1R) & End(f_1R) \end{pmatrix} \longrightarrow \begin{pmatrix} D & 0 \\ \mathbb{M}_{n\times 1}(D) & D' \end{pmatrix},$$

$$\begin{pmatrix} u_1 & 0 \\ u_2 & u_3 \end{pmatrix} \longmapsto \begin{pmatrix} u_1 & 0 \\ (\beta_k u_2)_{n\times 1} & \varphi(u_3) \end{pmatrix},$$

where D denotes $End(e_1R)$, is an isomorphism of rings. This completes the proof. ■

From now on, we will denote the formal triangular matrix ring

$$\begin{pmatrix} D & 0 \\ \mathbb{M}_{n\times 1}(D) & D' \end{pmatrix}$$

by (D, D^n, D').

Let $R = (D, D^n, D')$, where D is a division ring and D' is a division subring of $\mathbb{M}_n(D)$. Then $R = (D, 0, 0) \oplus (0, D^n, D')$, where $(D, 0, 0)$ is a simple right ideal and $(0, D^n, D')$ is a local right ideal with the maximal submodule $(0, D^n, 0)$.

Define $(D, \mathbb{M}_n(D))_i$ as the set of ordered pairs (a, A), where $a \in D$, $A \in \mathbb{M}_n(D)$, and a scalar multiplication

$$(a, A)(a_1, (x_k), A_1) = (aa_1 + A^{(i)}(x_k), AA_1),$$

where $(a_1, (x_k), A_1) \in R$ and $A^{(i)}$ denotes the ith row of A. It is easy to see that $(D, D')_i$ is a right R–submodule of $(D, \mathbb{M}_n(D))_i$. Also, if $B_i = \oplus_{j\neq i} e_j D$, where $\{e_1, \ldots, e_n\}$ is the natural basis for D^n over D, then $(D, D')_i$ is isomorphic to $\dfrac{(0, D^n, D')}{(0, B_i, 0)}$.

Lemma 2.2 $u : (D, D')_i \to (D, \mathbb{M}_n(D))_j$ is a nonzero R–homomorphism if and only if there exist $d_0 \in D$ and $A_0 \in \mathbb{M}_n(D)$ such that $A_0[j, k] = \delta_{ki}d_0$ $(k = 1, \ldots, n)$ and $u(d, A) = (d_0 d, A_0 A)$ for all $(d, A) \in (D, D')_i$.

Proof. Straightforward. ■

Lemma 2.3 *[3, Lemma 1] The property of having no (simple) middle class is inherited by factor rings.*

Lemma 2.4 *Suppose a ring $R = S \oplus T$ is a direct sum of two rings S and T, where S is semisimple. Then R has no (simple) middle class if and only if T has no (simple) middle class.*

Proof. Let M be an N–injective right R–module, where N is cyclic and nonsemisimple. Then N is isomorphic to a direct sum $S/A \oplus T/B$ of right R–modules for some right ideals A and B contained in S and T, respectively. Note that T/B is not semisimple (as both R– and T–modules). Since $M = MS \oplus MT$, MT is (T/B)–injective as both R– and T–modules. By assumption, MT is an injective right T–module. However, it is not difficult to see that it is also injective as an R–module. We may also show, in a similar way, that MS is an injective right R–module. This gives that M is an injective R–module. Thus we established the sufficiency part. The necessity is obvious by the above lemma. ■

Let S be a subring of a ring R and u a unit in R. Obviously, uSu^{-1} is a subring of R isomorphic to S as a ring. We call uSu^{-1} a *conjugate ring* of S in R.

Theorem 2.5 *Let R be a right nonsingular right Artinian ring. Then R has no right middle class if and only if $R \cong S \oplus T$, where S is a semisimple Artinian ring and T is zero or Morita equivalent to a formal triangular matrix ring of the form (D, D^n, D') where D is a division ring and D' is a division subring of $\mathbb{M}_n(D)$ such that for each conjugate ring U of D' in $\mathbb{M}_n(D)$, the set of i–th rows of elements in U span D^n as a left D–space for every $i = 1, \ldots, n$.*

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Proof. Assume first that R has no right middle class. By [3, Theorem 2], $R \cong S \oplus T$, where S is a semisimple Artinian ring and T is zero or a right Artinian right SI ring satisfying the properties $(P1)$ and $(P2)$. Suppose T is not zero. Then by Theorem 2.1, T is Morita equivalent to a formal triangular matrix ring of the form (D, D^n, D'), where D is a division ring and D' is a division subring of $\mathbb{M}_n(D)$. Since the property of having no right middle class is a Morita invariant property (as remarked, for example, in [3] before Proposition 5), the ring (D, D^n, D') has no right middle class. Note that $(D, 0)_1 \leq_e (D, \oplus_{j=1}^n De_{ij})_1$ for all $i = 1, \ldots, n$. By Lemma 2.2, $(D, De_{1k}D')_1$ is $(D, D')_k$–injective as a right R–module for all $k = 1, \ldots, n$. Since R has no right middle class and $(D, D')_k$ is nonsemisimple, $(D, De_{1k}D')_1$ must be injective for all $k = 1, \ldots, n$. On the other hand, we have $(D, 0)_1 \leq (D, De_{1k}D')_1 \leq (D, \oplus_{j=1}^n De_{1j})_1$, which gives that $De_{1k}D' = \oplus_{j=1}^n De_{1j}$ for all $k = 1, \ldots, n$. It therefore follows that the set of the k–th rows of elements of D' span D^n as a left D–space for all $k = 1, \ldots, n$. Now let $U = uD'u^{-1}$ for some unit u in $\mathbb{M}_n(D)$. Obviously, $R \cong (1, 0, u)R(1, 0, u^{-1}) = (D, D^n, U)$. Then (D, D^n, U) has no right middle class. Repeating the above arguments, we complete the proof of the necessity part.

For the sufficiency, it is enough, by Lemma 2.4, to show that if, for each conjugate ring U of D' in $\mathbb{M}_n(D)$, the set of i–th rows of elements in U span D^n as a left D–space for every $i = 1, \ldots, n$, then the ring (D, D^n, D') has no right middle class. Assume the contrary, i.e., assume that (D, D^n, D') has a right middle class. Note that the maximal right quotient ring of R is $Q = \mathbb{M}_{n+1}(D)$. By [3, Proposition 8], there exists $X \leq Q_R$ which contains the right socle of (D, D^n, D') properly such that $QX \neq Q$. Since the right socle of the ring (D, D^n, D') is $(D, D^n, 0)$, there exists a nonzero right D'–submodule Y of $\mathbb{M}_{(n+1)\times n}(D)$ such that

$$X = \begin{pmatrix} D & \\ D^n & Y \end{pmatrix}.$$

One can observe that if, for each $i = 1, \ldots, n$, there exists an element of Y (depending on i) whose i–th column has a nonzero entry, then $QX = Q$. Thus, there exists j such that the j–th column of each element of Y is zero. Without loss of generality, we may assume $j = 1$. Then there exist elements $d_1, \ldots, d_{n-1}$ of D which are not all zero such that $d_1A[2, 1] + \cdots + d_{n-1}A[n, 1] = 0$ for all $A \in D'$. We may choose $d_{n-1} \neq 0$. Let $B = e_{11} + (\sum_{i=2}^n e_{ii} + d_{i-1}e_{ni})$. Obviously, B is invertible in $\mathbb{M}_n(D)$, and all elements of $BD'B^{-1}$ have zero in the $(n, 1)$–th entry. It follows that the n–th rows of elements of $BD'B^{-1}$ cannot span D^n. This completes the proof. ∎

Corollary 2.6 *(i) Let D be a division ring and D' a division subring of D. Then the ring* $\begin{pmatrix} D & 0 \\ D & D' \end{pmatrix}$ *has no right middle class.*

(ii) Let D be a division ring and D' a division subring of $\mathbb{M}_2(D)$. Then the ring $R = (D, D^2, D')$ has no right middle class if and only if the set of the ith rows of elements of D' span D^2 for $i = 1, 2$.

Proof. (i) Clear by Theorem 2.5. (ii) If we take $n = 2$ in the proof of Theorem 2.5, then we deduce that when R has right middle class, there exists a nonzero element $d \in D$ such that $dA[2, 1] = 0$ for all $A \in D'$. This contradicts the fact that the set of the second rows of elements of D' span D^2. ∎

It is shown in [3, Proposition 6] that if R is a right Artinian right SI–ring with homogeneous right socle and a unique local module of length two up to isomorphism, then R has no right middle class. Example 2.9 below shows that the converse of this fact is not true in general. Before the example, we need the following proposition.

Proposition 2.7 *Let D be a division ring and D' be a division subring of $\mathbb{M}_n(D)$. Let $R = (D, D^n, D')$ and let $\mathfrak{R}$ be the set of the first rows of all elements of D'. Then R has a unique local right R–module of length two up to isomorphism if and only if $\bigcup_{(x_i) \in \mathfrak{R}} D(x_i) = D^n$.*

Proof. Let M be a local right R–module of length two. Then there exists an epimorphism $g : R \to M$, and since M is local, we must have $g(0, D^n, D') = M$. It follows that there exists a maximal submodule A of $(0, D^n, 0)$ such that $A = Ker(g)$. Thus the unique simple submodule of M, say N, is isomorphic to $(D, 0, 0)$.

Observe that $(D, \bigoplus_{j=1}^n De_{1j})_1$ is injective relative to $(D, D')_i$ for each $i = 1, \ldots, n$, by Lemma 2.2. Since R can be embedded into the sum $(D, 0, 0) \oplus (D, D')_1 \oplus \cdots \oplus (D, D')_n$, we get that $(D, \bigoplus_{j=1}^n De_{1j})_1$ is an injective R–module. It follows that $(D, \bigoplus_{j=1}^n De_{1j})_1$ is the injective hull of the local right R–module $(D, e_{11}D')_1$. There is an isomorphism from N to $(D, 0)_1$ which extends to a homomorphism f

from M into $(D, \bigoplus_{j=1}^{n} De_{1j})_1$. Since $(D, \bigoplus_{j=1}^{n} De_{1j})_1$ is nonsingular and M/N is singular, f must be a monomorphism. It therefore follows that every local right R–module of length two can be embedded into $(D, \bigoplus_{j=1}^{n} De_{1j})_1$.

Let M' be a local submodule of $(D, \bigoplus_{j=1}^{n} De_{1j})_1$ of length two. Then $M' = (D, X)_1$ for a right D'–subspace X of $\bigoplus_{j=1}^{n} De_{1j}$. Since M' is local, X must be one–dimensional as a D'–space. This shows that any local right R–submodule of length two in $(D, \bigoplus_{j=1}^{n} De_{1j})_1$ is of the form $(D, (\sum_{j=1}^{n} e_{1j}d_j)D')_1$ for some $d_1, \ldots, d_n \in D$. Moreover, one can also prove that there exists an isomorphism from $(D, (\sum_{j=1}^{n} e_{1j}d_j)D')_1$ onto $(D, e_{11}D')_1$ if and only if there exists a nonzero $d \in D$ such that $d(d_i)_{i=1}^{n} \in \mathfrak{R}$ if and only if there exists $d \in D$ and $(x_i)_{i=1}^{n} \in \mathfrak{R}$ such that $(d_i) = d(x_i)_{i=1}^{n}$ if and only if $(d_i)_{i=1}^{n} \in \bigcup_{(x_i) \in \mathfrak{R}} D(x_i)$. Now the result follows. ∎

Remark 2.8 *It can be easily seen from Corollary 2.6 (ii) that, for a division ring D and a division subring D' of $\mathbb{M}_2(D)$, the ring*

$$\begin{pmatrix} D & 0 \\ D & \\ D & D' \end{pmatrix}$$

has right middle class if and only if either all elements of D' are lower triangular matrices or all elements of D' are upper triangular matrices. Suppose, in particular, that all elements of D' are upper triangular matrices. Then for every $[a_{ij}] \in D'$, a_{22} is uniquely determined. Thus, we have a mapping from D' to D such that $[a_{ij}] \mapsto a_{22}$ for all $[a_{ij}] \in D'$ which is a ring monomorphism, that is, D' can be embedded into D as a ring.

Example 2.9 *(i) If D is a division ring and D' is a division subring of $\mathbb{M}_2(D)$ consisting only of lower triangular matrices, then the ring (D, D^2, D') is a right Artinian right SI ring which satisfies the properties $(P1)$ and $(P2)$. However, by Remark 2.8, (D, D^2, D') has right middle class. For instance, if we let δ be a derivation on D and consider the division subring*

$$D' = \left\{ \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix} \mid a \in D \right\}$$

of $\mathbb{M}_2(D)$, then the ring (D, D^2, D') has right middle class. Thus the converse of Theorem 2 of [3] is not true, in general.

(ii) Let $D = \mathbb{Z}_3$ and $w = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Observe that $D' = \{0, 1, w, w^2, \ldots, w^7\}$ is a field. By Remark 2.8, the ring (D, D^2, D') has no right middle class.

Example 2.10 *Let p be a prime integer, $F \leq F_1$ a field extension, and K a division subring of $\mathbb{M}_p(F)$ which properly contains the field of scalar matrices in $\mathbb{M}_p(F)$. Then the ring*

$$R = \begin{pmatrix} F_1 & 0 \\ F_1^p & K \end{pmatrix}$$

has no right middle class. If, in particular, we take $F = \mathbb{Q}$, then any division subring of $\mathbb{M}_p(\mathbb{Q})$ contains all scalar matrices. It follows that R has no right middle class for any division subring K of $\mathbb{M}_p(\mathbb{Q})$ which is not the field of scalar matrices.

Proof. We first claim that for any $i = 1, 2, \ldots, p$, the i–th rows of all elements of K span F^p as an F–space. To see this, let A be an element of K which is not a scalar matrix and let $m_A(x)$ be the minimal polynomial of A over F. Let $m_A(x) = u(x)v(x)$, where $u(x), v(x) \in F[x]$. Then $0 = m_A(A) = u(A)v(A)$, which implies that one of the determinants $\det(u(A))$ or $\det(v(A))$ is zero. Assume that $\det(u(A)) = 0$. Since K contains all scalar matrices over F, we have $u(A) \in K$. But K is a division ring which means that every nonzero matrix in K has nonzero determinant. This gives that $u(A) = 0$. Since $\deg(u(x)) \leq \deg(m_A(x))$ and $m_A(x)$ is the monic polynomial of least degree which assumes A as a root, we must have $u(x)$ and $m_A(x)$ are associates. It follows that $m_A(x)$ is irreducible over F. Then the characteristic polynomial $c_A(x)$ of A over F is a power of $m_A(x)$. This implies that $\deg(m_A(x))$ divides $\deg(c_A(x)) = p$. Since p is prime, $\deg(m_A(x))$ is either 1 or p. If $\deg(m_A(x)) = 1$, then A is similar to a scalar matrix, B say. In other words, there exists a $p \times p$ invertible matrix

P over F such that $P^{-1}AP = B$. Thus $A = PBP^{-1} = BPP^{-1} = B$, a contradiction. Therefore $m_A(x) \in F[x]$ is an irreducible polynomial of degree p.

Now we shall show that for any $i = 1, 2, \ldots, p$, the i–th rows of the matrices $I, A, \ldots, A^{p-1}$ span F^p as an F–space. In order to prove this, without loss of generality, we may choose $i = 1$. Assume the contrary, i. e., the first rows of the matrices $I, A, \ldots, A^{p-1}$ do not span F^p. Then the first rows of these matrices should be linearly dependent. So, there exist scalar matrices $c_0, c_1, \ldots, c_{p-1}$ over F, not all zero, such that the first row of the matrix $C = c_0 + c_1 A + \cdots + c_{p-1} A^{p-1}$ is zero. This gives that $\det(C) = 0$. Since C lies in K, we must have $C = 0$. But then A happens to be a root of a polynomial over F of degree at most $p - 1$, a contradiction. Consequently, for any $i = 1, 2, \ldots, p$, the i–th rows of all elements of K span F^p as an F–space. This gives that the F–space spanned by the i–th rows of elements of K contains the standard basis which is also contained in the F_1–space spanned by the i–th rows of elements of K. Therefore, for any $i = 1, 2, \ldots, p$, the i–th rows of all elements of K span F_1^p as an F_1–space. This fact is true for any conjugate of K in $\mathbb{M}_p(F)$ because, just as K, it also properly contains the field of scalar matrices over F. The proof is complete by Theorem 2.5. ∎

Remark 2.11 *Let p be a prime number and F be a field. Let $P(x)$ be an irreducible polynomial over F of degree p (if exists) and let $A \in \mathbb{M}_p(F)$ be such that $P(A) = 0$ (one can use the companion matrix of $P(x)$ from linear algebra to find such A). Then the set $K = \{c_0 + c_1 A + \cdots + c_{p-1} A^{p-1} : c_0, c_1, \ldots, c_{p-1} \in F\}$ is a field isomorphic to $F[x]/(P(x))$ and properly contains the field of scalar matrices over F. Indeed, we can show that all division subrings K of $\mathbb{M}_p(F)$ properly containing the field of scalar matrices are of this form: Let A be an element of K which is not a scalar matrix and let B be any nonzero element of K. Since the set of the first rows of $I, A, \ldots, A^{p-1}$ is a basis for F^p, we must have, by the same reasoning used in the proof of Example 2.10, there exist scalar matrices $c_0, c_1, \ldots, c_{p-1}, c_p$ over F, not all zero, such that $c_0 + c_1 A + \cdots + c_{p-1} A^{p-1} + c_p B = 0$. Here, clearly, $c_p \neq 0$. It follows that B is a linear combination of the powers A^i of A, where $i = 0, 1, \ldots, p - 1$. Therefore $K = \{c_0 + c_1 A + \cdots + c_{p-1} A^{p-1} : c_0, c_1, \ldots, c_{p-1} \in F\}$.*

Example 2.12 *Let F be a field and $P(x)$ be an irreducible polynomial over F of prime degree. Then the ring*

$$\begin{pmatrix} F & 0 \\ F[x]/(P(x)) & F[x]/(P(x)) \end{pmatrix}$$

has no right middle class.

Proof. Let $\deg(P(x)) = p$. Set $K = \{c_0 + c_1 A + \cdots + c_{p-1} A^{p-1} : c_0, c_1, \ldots, c_{p-1} \in F\}$, where A is the companion matrix of $P(x)$ over F. It is routine to check that the mapping

$$\begin{pmatrix} F & 0 \\ F[x]/(P(x)) & F[x]/(P(x)) \end{pmatrix} \longrightarrow \begin{pmatrix} F & 0 \\ F^p & K \end{pmatrix},$$

by

$$\begin{pmatrix} a & 0 \\ \sum_{i=0}^{p-1} c_i x^i & \sum_{i=0}^{p-1} d_i x^i \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 \\ \begin{matrix} c_0 \\ c_1 \\ \vdots \\ c_{p-1} \end{matrix} & \sum_{i=0}^{p-1} d_i A^i \end{pmatrix}$$

is a ring isomorphism. The result follows from Example 2.10 and Remark 2.11. ∎

Example 2.13 *The ring*

$$R = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \\ \mathbb{C} & C \end{pmatrix}, \text{ where } C = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \cong \mathbb{C},$$

has no right middle class. However, R has at least two nonisomorphic local right R–modules of length two by Proposition 2.7. Therefore, the converse of [3, Proposition 6] is not true, in general.

Note that if a ring of the form (D, D^n, D'), where D and D' are as above, has no right middle class, then the property of having no right middle class of (D, D^n, D') remains unaltered when we

replace D by any division ring containing D. However, such a replacement may result in an increased number of local modules of length two. On the other hand, as the following theorem shows, for the ring $R = (D, D^n, D')$, it is necessary that certain local R–modules of length two are isomorphic, which is also sufficient when $n = 2$.

Lemma 2.14 *Let $1 \le i, j \le n$. Then $(D, D')_i \cong (D, D')_j$ if and only if D' contains an element A such that $A[j, k] = \delta_{ik} c$ for some nonzero $c \in D$.*

Proof. The proof is straightforward by Lemma 2.2. ∎

Theorem 2.15 *Let D be a division ring and D' a division subring of $\mathbb{M}_n(D)$. If the ring*

$$R = \begin{pmatrix} D & 0 \\ \mathbb{M}_{n \times 1}(D) & D' \end{pmatrix}$$

has no right middle class, then there exists a conjugate D'' of D' in $\mathbb{M}_n(D)$ such that $(D, D'')_i \cong (D, D'')_j$ for each $i, j = 1, \ldots, n$ as right (D, D^n, D'')–modules. In particular, if $n = 2$, then R has no right middle class if and only if $(D, D')_1 \cong (D, D')_2$.

Proof. Since the ring R has no right middle class, by Theorem 2.5, D' has an element A such that $A[1, n] \neq 0$. By standard techniques of linear algebra, it is not difficult to see that there exists a unit y_1 in $\mathbb{M}_n(D)$ such that $B[1, i] = 0$ for all $i = 1, \ldots, n - 1$ and $B[1, n] = A[1, n]$, where $B = y_1 A y_1^{-1}$. Now, pick an element A' of D' such that $A'[1, n-1] \neq 0$. Then there exists a unit $y_2 \in \mathbb{M}_n(D)$ such that $B'[1, i] = 0$ for $1 \le i \le n$, $i \neq n - 1$, and $B'[1, n-1] = A'[1, n-1]$. Notice that conjugating B by y_2 does not effect the current form of B, i.e., $y_2 B y_2^{-1}$ is still a matrix whose $(1, n)$–th entry is nonzero and $(1, i)$–th entry is zero for every $i = 1, \ldots, n - 1$. If we continue in this fashion, we may find a unit y in $\mathbb{M}_n(D)$ such that the ring $y D' y^{-1}$ contains elements $A_2, \ldots, A_n$, where $A_i[1, k] = \delta_{ik} c_i$ for some $0 \neq c_i \in D$ for all $i = 2, \ldots, n$. Now the first part of the theorem follows by Lemma 2.14. Moreover, the remaining part also follows from the first part together with Corollary 2.6. ∎

Our aim in concluding this section is to complete our investigation of rings which have no right middle class. From Theorem 1 given in the introductory part, we know exactly what it means for a right SI–ring to have no right middle class . With the following theorem, we determine how precisely rings with no right middle class which are not right SI look like.

Theorem 2.16 *Let R be a ring which is not right SI. Then R has no right middle class if and only if $R \cong S \oplus \mathbb{M}_n(A)$, where S is a semisimple Artinian ring, n is a positive integer, and A is either zero or a local right Artinian ring whose Jacobson radical properly contains no nonzero ideals.*

Proof. The sufficiency follows from [9, Corollary 2.14]. For the necessity, suppose that R is a ring with no right middle class which is not right SI. By [3, Theorem 2], $R \cong S \oplus T$, where S is a semisimple Artinian ring and T is zero or it is a ring as in Theorem 2(iii) of [3]. If $T = 0$, then we are done. Let T be nonzero. Then by [3, Corollary 6], $T \cong \begin{pmatrix} \mathbb{M}_n(A) & 0 \\ X & B \end{pmatrix}$, where A is a (nonsemisimple) local right Artinian ring, B is a semisimple Artinian ring, and X is a B–$\mathbb{M}_n(A)$–bimodule. Note that T has no right middle class, too. As $J(A) \neq 0$, by [9, Corollary 2.14], we must have $X = 0$ and $J(A)$ properly contains no nonzero ideals. Since T is indecomposable, we must also have $B = 0$. This completes the proof. ∎

Corollary 2.17 *Let R be a right Noetherian ring. Then R has no right middle class if and only if $R \cong S \oplus T$, where S is a semisimple Artinian ring and T is zero or it is Morita equivalent to one of the following rings:*

(i) a right PCI–domain, or

(ii) a formal triangular matrix ring of the form (D, D^n, D') where D is a division ring and D' is a division subring of $\mathbb{M}_n(D)$ such that for each conjugate ring U of D' in $\mathbb{M}_n(D)$, the set of i–th rows of elements in U span D^n as a left D–space for every $i = 1, \ldots, n$, or

(iii) a local right Artinian ring whose Jacobson radical properly contains no nonzero ideals.

Proof. The sufficiency follows from [3, Proposition 5], Theorem 2.5, and Theorem 2.16. For the necessity, assume that R is a right Noetherian ring which has no right middle class. If R is right Artinian, then by [3, Theorem 2], Theorem 2.5, and 2.16, R is Morita equivalent to a ring which belongs to the class of rings in (ii) or (iii). If R is not right Artinian, then, by [3, Theorem 2], it is either Morita equivalent to a right PCI–domain or a V–ring with essential socle. Since R is right Noetherian, in the latter case R is semisimple Artinian. This completes the proof. ∎

Corollary 2.18 *Let R be any ring. Then R has no right middle class if and only if $R \cong S \oplus T$ where S is a semisimple Artinian ring and T is zero or it satisfies one of the following conditions:*

(i) T is Morita equivalent to a right PCI–domain, or

(ii) $T \cong \mathbb{M}_n(A)$, where n is a positive integer, and A is a local right Artinian ring whose Jacobson radical properly contains no nonzero ideals, or

(iii) T is an indecomposable right SI–ring with homogeneous essential right socle which satisfies one the following equivalent conditions (where Q is the maximal right quotient ring of T):

(a) Non-semisimple quasi–injective right T–modules are injective.

(b) Proper essential submodules of Q_T are poor.

(c) For any submodule A of Q_T containing $Soc(T_T)$ properly, $QA = Q$.

Proof. It follows from [3, Theorem 2 and Proposition 8] and Theorem 2.16. ∎

Remark 2.19 *Let R be a nonsemisimple right Artinian ring with no right middle class. For the sake of simplicity, we assume that R is indecomposable. We know that either R is a QF–ring with Jacobson radical square zero or R_R is poor. If R is a QF–ring, then it is not right SI (because a nonsemisimple right SI–ring with no right middle class cannot be right self–injective), and so, by Theorem 2.16, $R \cong \mathbb{M}_n(A)$ for some local right Artinian ring whose Jacobson radical properly contains no nonzero ideals. Then A is also QF. If I is a nonzero right ideal of R, then $J = r.ann(l.ann(I)) = I$. This gives that $cl(R_R) = 2$, i.e., R is a right chain ring with right composition length two. However, there are Artinian rings with no right (and left) middle class which are poor as a right (and left) module although they have right (or left) composition length two (see Example 2.20).*

Example 2.20 *Let F be the field $\mathbb{Q}(\sqrt{2})$ and $\alpha : F \longrightarrow \mathbb{Q}$ defined by $\alpha(a + b\sqrt{2}) = a$. Let $R = F \times F$ as additive abelian group and define the multiplication on R as*

$$(u,v)(w,z) = (uw, uz + v\alpha(w)).$$

Then R is a noncommutative local Artinian ring with $cl(R_R) = 3$ and $cl(_RR) = 2$. Since $cl(_RR) = 2$, $J(R)$ properly contains no nonzero ideals of R. It follows that R has no right (and left) middle class. Also, R does not satisfy the double annihilator condition for right ideals. Then R cannot be QF, i.e., both R_R and $_RR$ are poor.

Obviously, a ring with no right middle class is either right self–injective or poor as a right module over itself. It is known, from [3, Proposition 9], that the condition of Theorem 2(iii) in [3] are sufficient if the ring is taken to be right self–injective. The next two examples illustrate that these conditions are not sufficient in general even if the ring is poor. In particular, these examples also indicate that there are rings satisfying the conditions of Theorem 2(iii) in [3] which are not of the form $\mathbb{M}_n(A)$, where A is as in Theorem 2.16. We first need the following lemma.

Lemma 2.21 *Let R be a right semiartinian ring. Then R_R is poor if and only if R is not injective relative to a local right R–module of length two.*

Proof. The necessity is obvious. For the sufficiency, suppose that R is not injective relative to a local right R–module of length two. Let R be M–injective. Without loss of generality, we may choose M cyclic. Assume that M is not semisimple. Then there exists a local subfactor N of M of length two since M is semiartinian. This gives that R is N–injective, a contradiction. ∎

Example 2.22 *Let F be a field and V a finite dimensional vector space over F of dimension greater than 1. Let $R = \left\{ \begin{pmatrix} a & 0 \\ v & a \end{pmatrix} : a \in F, \ v \in V \right\}$. Then R is a commutative local Artinian ring which satisfies the conditions of Theorem 2(iii) in [3]. However, R has right middle class by [9, Corollary 2.14]. Note too that R is poor as a module over itself. Indeed, if L is a local module of length two with simple submodule S, then there is an isomorphism from S into R which cannot be extended to a homomorphism from L. Otherwise, R would contain a local module of length two, which is impossible since $\dim_F(V) > 1$. Then R is not L–injective. Hence, by Lemma 2.21, R is poor.*

Example 2.23 *Let* $R = \begin{pmatrix} \mathbb{Z}/4\mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$. *Then*

(1) $Soc(R_R) = J(R) = Z(R_R)$,

(2) R *has essential homogeneous right socle,*

(3) *there is a unique noninjective simple right R–module up to isomorphism,*

(4) R_R *is poor, and*

(5) R *has right middle class.*

Proof. (1) It follows from [5, Proposition 4.2] that $Soc(R_R) = J(R) = Z(R_R) = \begin{pmatrix} 2\mathbb{Z}/4\mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 \end{pmatrix}$

(2) Since R is right Artinian, $Soc(R_R) \leq_e R_R$. On the other hand, $Soc(R_R) = \begin{pmatrix} 2\mathbb{Z}/4\mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix} \oplus$
$\begin{pmatrix} 0 & 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 \end{pmatrix}$, where simple summands are isomorphic.

(3) Since $R/J(R) \cong S \oplus S'$, where

$$S = \begin{pmatrix} \mathbb{Z}/2\mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } S' = \frac{\begin{pmatrix} 0 & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}}{\begin{pmatrix} 0 & 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 \end{pmatrix}},$$

any simple right R–module is isomorphic to either S or S'. As $Soc(R_R) \leq_e R_R$, S_R cannot be injective. To establish (3), we shall show that S'_R is injective. Note that all proper essential right ideals of R are

$$I_1 = \begin{pmatrix} 2\mathbb{Z}/4\mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 \end{pmatrix}, \ I_2 = \begin{pmatrix} \mathbb{Z}/4\mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 \end{pmatrix}, \text{ and } I_3 = \begin{pmatrix} 2\mathbb{Z}/4\mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}.$$

If $f : I_1 \longrightarrow S$ is a nonzero homomorphism of R–modules, then $f(I_1) = S$. But $I_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$ while $S \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$, a contradiction. Thus $Hom_R(I_1, S) = 0$. Similarly, $Hom_R(I_2, S) = 0$. Now, let $f : I_3 \longrightarrow S$ be a nonzero homomorphism. Observe that there are only two maximal right R–submodule of $I_3 : Soc(R_R)$ and $M = \begin{pmatrix} 0 & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$. Since $I_3/M \not\cong S$, we must have $Ker(f) = Soc(R_R)$. It follows that there exists a unique nonzero homomorphism $f : I_3 \longrightarrow S$, which can be extended to a homomorphism $R \longrightarrow S$. Therefore, S_R is injective.

(4) Note that there are two local right R–modules of length two up to isomorphism: $X = \begin{pmatrix} \mathbb{Z}/4\mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}$. Indeed, we can decompose R_R as $R = X \oplus Y$. Notice that both X and Y are local right R–modules of length two. Now let M be a local right R–module of length two. Then there exists an epimorphism $f : R \longrightarrow M$. Since M is local, $f(X) = M$ or $f(Y) = M$. This gives that $X \cong M$ or $Y \cong M$.

R is not X–injective because the map $\begin{pmatrix} 2\mathbb{Z}/4\mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow R$ defined by $\begin{pmatrix} \overline{2} & 0 \\ 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} \overline{2} & 0 \\ x & 0 \end{pmatrix}$, where $0 \neq x \in \mathbb{Z}/2\mathbb{Z}$, is an R–homomorphism which does not extend to a homomorphism $f : X \longrightarrow R$. Indeed, if $f \begin{pmatrix} \overline{1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ y & b \end{pmatrix}$, then $f \begin{pmatrix} \overline{2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}$, a contradiction.

Now we claim that R is not Y–injective. To see this, let $f : \begin{pmatrix} 0 & 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 \end{pmatrix} \longrightarrow R$ with $\begin{pmatrix} 0 & 0 \\ \overline{1} & 0 \end{pmatrix} \longmapsto$
$\begin{pmatrix} \overline{2} & 0 \\ \overline{1} & 0 \end{pmatrix}$. Then f is a homomorphism which cannot extend to a homomorphism $g : Y \longrightarrow R$. Indeed, if $g \begin{pmatrix} 0 & 0 \\ 0 & \overline{1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ y & b \end{pmatrix}$, then $a = y = 0$ since $\begin{pmatrix} 0 & 0 \\ 0 & \overline{1} \end{pmatrix} \begin{pmatrix} \overline{1} & 0 \\ 0 & 0 \end{pmatrix} = 0$ and $\begin{pmatrix} a & 0 \\ y & b \end{pmatrix} \begin{pmatrix} \overline{1} & 0 \\ 0 & 0 \end{pmatrix} =$
$\begin{pmatrix} a & 0 \\ y & 0 \end{pmatrix}$. Thus, $g \begin{pmatrix} 0 & 0 \\ 0 & \overline{1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$. But we also have $\begin{pmatrix} 0 & 0 \\ 0 & \overline{1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \overline{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \overline{1} & 0 \end{pmatrix}$ and
$\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \overline{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \overline{b} & 0 \end{pmatrix}$, a contradiction.

By Lemma 2.21, R_R is poor.

(5) By [9, Corollary 2.14], R has right middle class. ∎

3   Artinian Rings With No Simple Middle Class

Notice that right V–rings have automatically no simple middle class. In the theorem below, we consider right GV–rings without simple middle class whose proof uses almost the same arguments as those used in the proof of [3, Lemma 8]. Before giving the theorem, we need the following lemma.

Lemma 3.1 *Suppose that R is not a right V–ring. Then R is a right GV–ring with no simple middle class if and only if R has a simple projective poor module.*

Proof. It follows from [1, Corollaries 4.4 and 4.5]. ∎

As a direct consequence of Lemma 3.1, we have the following lemma.

Lemma 3.2 *Suppose that a ring R has no simple middle class. Then R is a right GV–ring or every simple projective right module is injective.*

The proof of the following theorem uses the notion of orthogonal modules. Recall that two modules are said to be orthogonal if they have no nonzero isomorphic submodules.

Theorem 3.3 *Suppose that R is a right nonsingular right GV–ring which is not a right V–ring. If R has no simple middle class, then there is a ring decomposition $R = S \oplus T$, where S is semisimple Artinian and the right socle of T is nonzero poor homogeneous. If, further, R_R has finite uniform dimension, then the converse also holds.*

Proof.   Claim 1 : $Soc(R_R)$ does not contain a direct sum of two infinitely generated orthogonal submodules.

Assume that A and B are infinitely generated orthogonal submodules of $Soc(R_R)$. Then A and B are noninjective modules. Let f be a homomorphism from $E(B)$ into $E(A)$. We will show that $Ker(f) \leq_e E(B)$. Let X be a nonzero submodule of $E(B)$. Then $X \cap B \neq 0$. If we assume that $Ker(f) \cap X = 0$, then we get $0 \neq X \cap B \cong f(X \cap B)$. But this contradicts the fact that A and B are orthogonal. Hence, $Ker(f) \cap X \neq 0$. It follows that $E(B)/Ker(f) \cong Im(f)$ is singular, and hence $Im(f) = 0$. Thus, A is $E(B)$–injective. Since A is poor by [1, Corollary 4.5], B is injective, which is a contradiction.

Claim 2 : One of the two nonisomorphic simple right ideals is injective.

If S_1 and S_2 are two nonisomorphic noninjective simple right ideals, then S_1 is $E(S_2)$–injective which implies by assumption that either $E(S_2)$ is semisimple or S_1 is injective. This gives that either S_1 or S_2 is injective.

By the same technique above, one can observe that a simple right ideal which is orthogonal to an infinitely generated semisimple right ideal is injective. Thus, $Soc(R_R)$ can have only finitely many homogeneous components. Let $H_1, H_2, \ldots, H_n$ be the homogeneous components of $Soc(R_R)$. Notice that all the H_i's will have to be injective except possibly for at most one of them. If $n = 1$ and H_1 is noninjective, then $Soc(R_R)$ is homogeneous and poor, and so we are done. Now suppose $n > 1$, H_1 is either noninjective or zero, and $H_2, \ldots, H_n$ are injective. Set $S = H_2 \oplus \cdots H_n$. Then $R = S \oplus T$ for some right ideal T.

Claim 3 : $R = S \oplus T$ is a ring direct sum, where $Soc(T_T)$ is nonzero poor homogeneous.

Obviously, $TS = 0$. If $ST \neq 0$, then there exists $s \in S$ such that $sT \neq 0$. But $Soc(T_T) \leq ann_r(s)$ and $ST \leq S$. We will show that $X = ann_r(s) \cap T \leq_e T$. Let $I \leq T$ such that $X \cap I = 0$. Define $f : I \to R$, $x \mapsto sx$. Clearly $I \cong Im(f) = sI$, and hence $I = 0$. Therefore, $sT \cong T/(ann_r(s) \cap T)$ is zero, which gives that $ST = 0$. Thus, we obtain a ring decomposition $R = S \oplus T$, where S is semisimple Artinian and $Soc(T_T) \cong H_1$ is homogeneous, and poor if H_1 is nonzero. It is routine to check that T is a right GV–ring, too. If $Soc(T_T)$ is zero, then T has to be a right V–ring. But this leads to the fact that R is a right V–ring, a contradiction.

For the last statement, suppose that R_R has finite uniform dimension and $Soc(T_T)$ is poor homogeneous. Then $Soc(T_T)$ is a direct sum of finitely many isomorphic simple right ideals. This gives that simple right ideals of T are poor, by assumption. On the other hand T is also a GV–ring. Since a simple module is either projective or singular and a simple projective right T–module is isomorphic to a simple right ideal of T, we get that T has no simple middle class. Now the result follows from Lemma 2.4. ∎

Corollary 3.4 *Let R be a nonsemisimple ring with no simple middle class. If R is a right semiartinian right GV–ring which is not a right V–ring, then we have a ring decomposition $R = S \oplus T$, where S is semisimple Artinian and $Soc(T_T)$ is poor homogeneous.*

Proof. Since right semiartinian right GV–rings are nonsingular the result follows from Theorem 3.3. ∎

Corollary 3.5 *[1, Theorem 4.7] Let R be a semiperfect right GV–ring. If R has no simple middle class, then we have a ring decomposition $R = S \oplus T$, where S is semisimple Artinian and T is a semiperfect ring with homogeneous projective and poor right socle.*

Proof. Since semilocal right GV–rings are right semiartinian, the proof follows from Corollary 3.4. ∎

Proposition 3.6 *Let R be a right semiartinian ring with a singular right socle. If R has no simple middle class, then R is an indecomposable ring with unique noninjective simple R–module up to isomorphism and $Soc(R_R)$ is homogeneous.*

Proof. Since $Soc(R_R) \neq 0$, R is not a GV–ring. If a simple right ideal is injective, then it is projective. But this is a contradiction since simple right ideals are singular. Consequently, every simple right ideal of R is noninjective. Let S be a simple right ideal and M be any noninjective simple singular right R–module. Since R is right semiartinian, there exists $N \leq E(M)$ such that M is maximal in N. Since S is poor, there exists a homomorphism $f : N \to E(S)$ such that $f(N) \nsubseteq S$. Then $S \subsetneq f(N)$. Since N has composition length two whereas $f(N)$ has at least two, f must be monic. Thus, $S \cong M$. Noninjective simple modules are not projective because of Lemma 3.2, whence they are singular. Thus, R has a unique noninjective simple module up to isomorphism. Moreover, since $Soc(R_R)$ is a direct sum of noninjective simple singular right ideals, it is homogeneous. Also, it is clear that a semiartinian ring with homogeneous socle is indecomposable. ∎

Theorem 3.7 *If R is a right Artinian nonsemisimple ring with no simple middle class, then R has a ring decomposition $R = S \oplus T$, where S is semisimple Artinian and $Soc(T_T)$ is poor homogeneous. Moreover, $Soc(R_R)$ is either projective or singular.*

Proof. If R is a right GV–ring, then we are done with by Theorem 3.3. Suppose that R is not a right GV–ring. Then every simple projective module is injective by Lemma 3.2. Write $Soc(R_R) = P \oplus N$, where P is the sum of all simple projective right ideals of R. Then P is injective. Therefore, $R = P \oplus K$ for some right ideal K. Since N does not have any simple projective summand, P is an ideal of R. Now we will show that $R = P \oplus K$ is a ring direct sum. Obviously, $KP = 0$. Assume that $PK \neq 0$. Then there exists $p \in P$ such that $pK \neq 0$. But $Soc(K) \subseteq ann_r(p)$. We claim that $X = ann_r(p) \cap K \leq_e K$. Let $I \leq K$ such that $X \cap I = 0$. Define $f : I \to R$ such that $x \mapsto px$ for all $x \in I$. Then $I \cong Im(f) = pI$, and hence $I = 0$. Note that P is nonsingular since it is semisimple projective. It follows that $pK \cong K/(ann_r(p) \cap K)$ is both singular and nonsingular, and hence $PK = 0$. Thus, $R = P \oplus K$ is a ring direct sum. Then $K \cong R/P$ has no simple middle class by Lemma 2.3. K is nonzero because R is nonsemisimple. Hence, $Soc(K)$ is nonzero, too. Also, $Soc(K) \cong N$ is singular. By Proposition 3.6, K is an indecomposable ring with singular poor homogeneous right socle. ∎

Theorem 3.8 *Let R be a right Artinian ring. Then R has no simple middle class if and only if there is a ring decomposition $R = S \oplus T$ where S is semisimple Artinian and T is zero or has one of the following properties:*
    (1) T is a right SI–ring with homogeneous right socle.
    (2) T has a unique noninjective simple right R–module up to isomorphism, and the right socle of T is (homogeneous) singular.

Proof. ($\Rightarrow$) It follows from Lemma 3.2, Theorem 3.3, Proposition 3.6 and Theorem 3.7.
    ($\Leftarrow$) Let T be a nonzero ring which is not a V–ring and assume that it satisfies (1). Since T is right Artinian, we have a decomposition $T = e_1T \oplus \ldots \oplus e_nT \oplus f_1T \oplus \ldots \oplus f_kT$, where e_i and f_j form a complete set of local orthogonal idempotents, e_iT are isomorphic simple right ideals, and f_jT are nonsimple local T–modules. Since T is right SI, the simple modules of the form f_jT/f_jJ are injective, where J denotes the Jacobson radical of T. Therefore, if a right module does not contain an isomorphic copy of e_iT, then it is semisimple. Now assume that e_iT is M–injective, where M is a

cyclic right module. Then we have a decomposition $M = A_1 \oplus \ldots \oplus A_p \oplus B_1 \oplus \ldots \oplus B_q$, where A_k and B_t are indecomposable modules such that the A_k's do not contain an isomorphic copy of e_iT and the B_t's contain an isomorphic copy of e_iT. By the above argument, $A_1 \oplus \ldots \oplus A_p$ is semisimple. On the other hand, e_iT is B_t–injective. One can observe that $B_1 \oplus \ldots \oplus B_q$ is semisimple, too. Hence, e_iT is poor.

Now assume that T satisfies (2). By assumption, T has no simple projective module. Then we have a decomposition $T = f_1T \oplus \ldots \oplus f_mT$, where f_iT are nonsimple local modules. Let f_iT/f_iJ be a noninjective module for some i. Assume that f_iT/f_iJ is M–injective for a cyclic module M. We can write $M = A_1 \oplus \ldots \oplus A_p \oplus B_1 \oplus \ldots \oplus B_q$, where A_k and B_t are indecomposable modules such that the A_k's do not contain an isomorphic copy of f_iT/f_iJ and the B_t's contain an isomorphic copy of f_iT/f_iJ. Because T has a unique noninjective simple module up to isomorphism, $A_1 \oplus \ldots \oplus A_p$ is semisimple. $B_1 \oplus \ldots \oplus B_q$ is also semisimple since f_iT/f_iJ is B_t–injective for each $t = 1, \cdots, q$. Hence, T has no simple middle class. Now, the theorem follows from Lemma 2.4. ∎

Following [1], we call a ring R simple–destitute if every simple right R–module is poor. Notice that, just as V–rings, simple–destitute rings also constitute a natural subclass of rings with no simple middle class. In [1, Theorem 5.2], it is proved that if a right Artinian ring R has only one simple module up to isomorphism, then R is simple–destitute. Now we establish the converse of this theorem as follows.

Corollary 3.9 *Assume that R is a right Artinian ring. R is simple–destitute if and only if either R is semisimple or R has a unique simple module up to isomorphism.*

Proof. ($\Leftarrow$) It follows from [1, Theorem 5.2].
($\Rightarrow$) If R is semisimple, then we are done. Suppose R is not semisimple. It follows from [1, Theorem 5.3] that $Soc(R_R)$ is singular. Then R is an indecomposable ring by Proposition 3.6. Since R is neither a right V–ring nor a right SI–ring, we get the desired result by Theorem 3.8. ∎

We see, in [1] and [3], that the ring $S = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix}$, where F is a field, is of a particular interest. In [1], it is shown that S has no simple middle class. In [3], Er et al. proved that S has, indeed, no right middle class. It is also proved, in [3], that a QF–ring R with $J(R)^2 = 0$ and homogeneous right socle has no right middle class. In the following theorem, we give a more general result by replacing QF with Artinian serial. Note that the class of Artinian serial rings contains that of both QF–rings of above type and rings in the form of S.

Theorem 3.10 *If R is an Artinian serial ring with $J(R)^2 = 0$ and homogeneous right socle, then R has no (simple) middle class.*

Proof. Since R is Artinian serial, we can write $R = \oplus_{i=1}^n e_iR$, where e_i's are local idempotents and e_iR's are uniserial. Suppose e_kR is not simple for some $k = 1, \ldots, n$. Since $e_kJ(R)$ is the unique maximal submodule of e_kR, $e_kJ(R) = l.ann_{e_kR}(J(R)) = Soc(e_kR)$. Moreover, e_kR is an injective R–module by [2, 13.5, p.124]. It follows that, for each $i = 1, \ldots, n$, e_iR is either a simple module or an injective local module of length two. Now let e_tR and $e_{t'}R$ be nonsimple. By homogeneity of the right socle, we have $Soc(e_tR) \cong Soc(e_{t'}R)$. Then the injectivity of $e_{t'}R$ yields an isomorphism between e_tR and $e_{t'}R$. Thus the nonsimple e_iR's are all isomorphic to each other.

Now let M be a (simple) module. Assume M is N–injective, where N is cyclic. Since R is an Artinian serial ring, by [4, Theorem 5.6], $N = \oplus_{k=1}^r N_k$, where N_k's are cyclic uniserial. If N is not semisimple, then there exists t such that N_t is not simple. Since N_t is cyclic and local, $N_t \cong e_jR$ for some $j = 1, \ldots, n$. This gives that M is e_jR–injective. Also, M is injective relative to any e_iR which is simple. It follows that M is R–injective, i.e., it is injective. This completes the proof. ∎

Corollary 3.11 *Let R be an indecomposable Artinian serial ring. Then R has no right middle class if and only if $J(R)^2 = 0$ and R has homogeneous right socle.*

Theorem 3.10 shows that, for a nonsemisimple Artinian serial ring R with $J(R)^2 = 0$ and homogeneous right socle, R has no right middle class if and only if R has no simple middle class. However, one can find an Artinian serial ring with homogeneous right socle and no simple middle class which has right middle class, as the following example illustrates.

Example 3.12 *Let $R = \mathbb{Z}/8\mathbb{Z}$. Then R is an Artinian chain ring with no simple middle class by Corollary 3.9. However, R has right middle class since $J^2(R) \neq 0$.*

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4 Commutative Rings

In this section, we focus on commutative rings and investigate the property of having no (simple) middle class. We see that commutative rings with no middle class are precisely those Artinian rings which decompose into a sum of a semisimple ring and a ring of composition length two. We start with the following lemma.

Lemma 4.1 *[5, Exercise 17, Ch. 1, Sec. B] Let R be a commutative Noetherian ring, and let P, Q be prime ideals of R. Then $P \subseteq Q$ if and only if $Hom_R(E(R/P), E(R/Q)) \neq 0$.*

Proposition 4.2 *If R is a commutative Noetherian ring with no middle class, then R is Artinian.*

Proof. Let R be a commutative Noetherian ring with no middle class. We shall complete the proof by showing that Krull dimension of R is zero, i.e., every prime ideal of R is maximal. If R is a V–ring, then there is nothing to prove. So, assume that R is not a V–ring. Then there exists a maximal ideal P of R such that R/P is not injective. Hence $E(R/P)$ is not semisimple. Let Q be any prime ideal of R such that $Q \neq P$. Then by above lemma, $Hom_R(E(R/P), E(R/Q)) = 0$, and so R/Q is injective relative to $E(R/P)$. Since R has no middle class and $E(R/P)$ is nonsemisimple, R/Q is injective. Thus R/Q is a self–injective domain, which implies that Q is a maximal ideal. This completes the proof. ■

Theorem 4.3 *A commutative ring R has no middle class if and only if there is a ring decomposition $R = S \oplus T$, where S is a semisimple Artinian ring, and T is zero or a local ring whose maximal ideal is minimal.*

Proof. Suppose first that R has no middle class. Then, by [3, Theorem 2], there is a ring decomposition $R = S \oplus T$ where S is semisimple Artinian and T is either zero or fits in one of the following three cases:

Case I : T is Morita equivalent to a right PCI domain T'. In this case, since T' is right Noetherian, so is T. Then by Proposition 4.2, T is Artinian. Thus T' is an Artinian domain, and hence a simple ring. This gives that T is also a simple ring. Since T is commutative, it is a field.

Case II : T is an indecomposable SI–ring which is either Artinian or a V–ring. Assume first that T is Artinian. Then T is a finite product of local rings. Thus indecomposability gives that T is a commutative local Artinian ring. Suppose that T is not a field. Then there is a minimal nonzero ideal A of T. Notice that $A \cong T/J(T)$. But since T is an SI–ring and $T/J(T)$ is singular as a T–module, A is injective. Then A is a direct summand of T, which contradicts the indecomposability of T. Therefore T is a field. Now let T be a V–ring. We may assume that T is not Noetherian. Then by [3, Lemma 5], T is semiartinian. This gives that $soc(T) \neq 0$. Let A be a nonzero minimal ideal of T. Since $J(T) = 0$, there exists a maximal ideal $\mathfrak{M}$ which does not contain A. Then $A \oplus \mathfrak{M} = T$. It follows that $A = T$ and hence T is a field.

Case III : T is an indecomposable Artinian ring with $soc(T) = J(T)$. Note that, just as with Case II, R is a local ring. It, therefore, follows from [9, Corollary 2.14] that T is a ring whose maximal ideal $J(T)$ is minimal.

Conversely, if T is a commutative local ring whose maximal ideal is simple, then, clearly, T has a unique (up to isomorphism) local module of length two (which is, indeed, T itself), and has homogeneous $soc(T) = J(T)$. Thus, by [3, Proposition 7], T has no middle class. Now the result follows from Lemma 2.4. ■

We give the following immediate consequences of the above theorem.

Corollary 4.4 *Any commutative ring with no middle class is Artinian.*

Corollary 4.5 *A commutative ring R is a local ring whose (unique) maximal ideal is minimal if and only if*
(i) R is indecomposable Artinian,
(ii) $soc(R) = J(R)$, and
(iii) R has no middle class.

Now we turn our attention to commutative Noetherian rings with no simple middle class although they need not be Artinian as the following lemma shows.

Lemma 4.6 *A commutative local ring (not necessarily Noetherian) has no simple middle class.*

Proof. Let R be a commutative local ring with the unique maximal ideal $\mathfrak{M}$. Let $R/\mathfrak{M}$ be (R/I)–injective for some proper ideal I of R. Then R/I is a V–ring since its unique simple module is injective. This gives that $I = \mathfrak{M}$, and so $R/\mathfrak{M}$ is a poor module. This completes the proof. ■

Theorem 4.7 *Let R be a commutative Noetherian ring. Then R has no simple middle class if and only if there is a ring decomposition $R = S \oplus T$ where S is semisimple Artinian and T is a local ring.*

Proof. The sufficiency follows easily from Lemma 2.4 together with the above lemma. For the necessity, let R be a commutative Noetherian ring with no simple middle class. Suppose R is not semisimple Artinian. Then R is not a V–ring, and so there exists a maximal ideal $\mathfrak{M}$ of R such that $R/\mathfrak{M}$ is a poor R–module. Let $\mathfrak{P}$ be a prime ideal of R with $\mathfrak{P} \nsubseteq \mathfrak{M}$. Then $R/\mathfrak{M}$ is $E(R/\mathfrak{P})$–injective by Lemma 4.1. Then $E(R/\mathfrak{P})$ is semisimple, i.e., $E(R/\mathfrak{P}) = R/\mathfrak{P}$ and $\mathfrak{P}$ is a maximal ideal of R. Since $R/\mathfrak{P}$ is injective, by [8, Theorem 3.71], $R_\mathfrak{P}$ is a field. This, in particular, gives that $\mathfrak{P}$ contains no prime ideals properly, and that $\mathfrak{P}^k = \mathfrak{P}$ for every positive integer k. Since R is Noetherian, there exist minimal prime ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ of R such that $\mathfrak{P}_1^{t_1} \cap \ldots \cap \mathfrak{P}_n^{t_n} = 0$. If R is local, then we are done. So, suppose R is not local. If $\mathfrak{P}_i$ is contained in $\mathfrak{M}$ for every $i = 1, \ldots, n$, then for any maximal ideal $\mathfrak{P}$ of R, $\mathfrak{P}_1^{t_1} \cap \ldots \cap \mathfrak{P}_n^{t_n} = 0 \subseteq \mathfrak{P}$ which yields $\mathfrak{P}_j \subseteq \mathfrak{P}$ for some j. Then we must have, by above arguments, $\mathfrak{P}_j = \mathfrak{P} = \mathfrak{M}$, a contradiction. Thus we may arrange the $\mathfrak{P}_i$'s in such a way that $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ are not contained in $\mathfrak{M}$ but $\mathfrak{P}_{s+1}, \ldots, \mathfrak{P}_n$ are, for some $s < n$. It follows that $\mathfrak{P}_1, \ldots, \mathfrak{P}_s, \mathfrak{M}$ is the complete list of all maximal ideals of R, and that $\mathfrak{P}_1 \cap \ldots \cap \mathfrak{P}_s \cap \mathfrak{P}_{s+1}^{t_{s+1}} \cap \ldots \cap \mathfrak{P}_n^{t_n} = 0$. It is also easy to see that $(\mathfrak{P}_1 \cap \ldots \cap \mathfrak{P}_s) \oplus (\mathfrak{P}_{s+1}^{t_{s+1}} \cap \ldots \cap \mathfrak{P}_n^{t_n}) = R$. Notice that $\mathfrak{P}_{s+1}^{t_{s+1}} \cap \ldots \cap \mathfrak{P}_n^{t_n}$ is a semisimple Artinian ring isomorphic to $R/(\mathfrak{P}_1 \cap \ldots \cap \mathfrak{P}_s)$ whereas $\mathfrak{P}_1 \cap \ldots \cap \mathfrak{P}_s$ is a local ring isomorphic to $R/(\mathfrak{P}_{s+1}^{t_{s+1}} \cap \ldots \cap \mathfrak{P}_n^{t_n})$. This completes the proof. ■

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